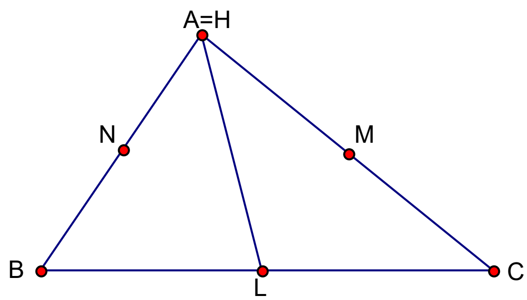
**Inequality involving triangles**
https://www.linkedin.com/groups/8313943/8313943-6386243001064189956
In triangle $ABC$, if $L,M,N$ are midpoints of $AB,AC,BC$, and $H$ is orthogonal center of triangle $ABC$. Prove that

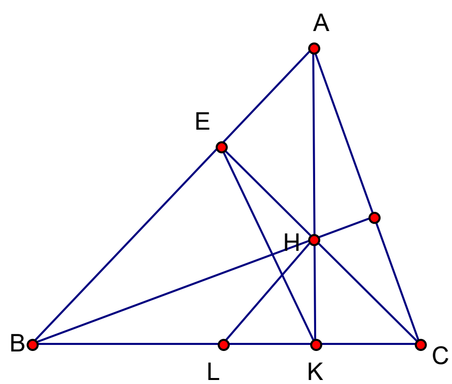
$$LH^2+MH^2+NH^2 \le (1/4)(AB^2+AC^2+BC^2).$$

**Solution by Arkady Alt , San Jose, California, USA.**

We will prove that inequality of the problem holds if $\triangle ABC$ is non-obtuse triangle.
If $\triangle ABC$ is right triangle (let $\angle A = 90^\circ$ ) then orthocenter $H$ coincide with $A$ and $HL = AL = R = a/2, HN = c/2, HM = b/2$ and,therefore,
$LH^2+MH^2+NH^2 = \frac{1}{4}(a^2+b^2+c^2) = (1/4)(AB^2+AC^2+BC^2)$




Let $\triangle ABC$ is an acute triangle.




Since* $HB = 2R\cos B$, $BL = a/2 = R\sin A$ and $\angle HBL = 90^\circ - C$ then $\cos\angle HBL = \sin C$ and
by Cosine Theorem we obtain $LH^2 = HB^2 + LB^2 - 2HB \bullet LB\sin C =$
$4R^2\cos^2 B + R^2\sin^2 A - 4R^2\sin A\cos B\sin C = R^2(4\cos^2 B + \sin^2 A - 4\sin A\cos B\sin C) =$
$R^2(\sin^2 A - 4\cos B(\sin A\sin C - \cos B))$. Noting that $\sin A\sin C - \cos B =$
$\sin A\sin C + \cos(A+C) = \cos A\cos C$ we obtain $LH^2 = R^2(\sin^2 A - 4\cos A\cos B\cos C)$.
Therefore, $LH^2+MH^2+NH^2 = R^2\sum(\sin^2 A - 4\cos A\cos B\cos C) =$
$R^2(\sin^2 A + \sin^2 B + \sin^2 C - 12\cos A\cos B\cos C)$ and original inequality can be equivalently rewritten as

$R^2(\sin^2 A + \sin^2 B + \sin^2 C - 12\cos A\cos B\cos C) \le R^2(\sin^2 A + \sin^2 B + \sin^2 C) \iff$
$0 \le \cos A\cos B\cos C.$

* $\triangle KEB$ is similar to $\triangle ACB$ with coefficient of homothety $|\cos B|$ and $BH$ is diameter of the circumcircle of $\triangle KEB$.